

ON THE ORDERS OF ARC-TRANSITIVE GRAPHS

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Dedicated to the memory of Ákos Seress

ABSTRACT. A graph is called *arc-transitive* (or *symmetric*) if its automorphism group has a single orbit on ordered pairs of adjacent vertices, and 2-arc-transitive if its automorphism group has a single orbit on ordered paths of length 2. In this paper we consider the orders of such graphs, for given valency. We prove that for any given positive integer k , there exist only finitely many connected 3-valent 2-arc-transitive graphs whose order is kp for some prime p , and that if $d \geq 4$, then there exist only finitely many connected d -valent 2-arc-transitive graphs whose order is kp or kp^2 for some prime p . We also prove that there are infinitely many (even) values of k for which there are only finitely many connected 3-valent symmetric graphs of order kp where p is prime.

1. INTRODUCTION

A graph is called *arc-transitive* (or *symmetric*) if its automorphism group has a single orbit on the set of all ordered pairs of adjacent vertices in the graph. The study of such graphs has a long and interesting history, highlighted at an early stage by ingenious work by Tutte [24, 25] on the cubic (3-valent) case.

This paper concerns the orders of finite symmetric graphs of given valency.

Vertex-transitive graphs of prime order were shown to be circulants (Cayley graphs for cyclic groups) by Turner [23], and then those which are symmetric were determined by Chao [2]. A few years later, Cheng and Oxley [3] found all symmetric graphs of order $2p$ for p prime. (In fact Cheng and Oxley classified all graphs of order $2p$ that are both vertex- and edge-transitive, and proved that all of these graphs are symmetric.)

More recently, numerous papers have been published in which the authors classify all symmetric graphs with given small valency (usually 3, 4 or 5) and with order of the form kp or kp^2 for a fixed integer k and variable prime p (see [10, 11, 12] for example), and we are aware of a number of other attempts to achieve such classifications, using voltage graphs and more general covering techniques. In many of these papers, the authors show that for a given k , there can be only finitely many such graphs.

We will show that this is always true when we restrict our attention to the case where the graph is 2-arc-transitive (meaning that its automorphism group has a single

orbit on the set of all ordered paths (u, v, w) of length 2) and has given valency greater than 3, as well as in the case of 2-arc-transitive cubic graphs of order kp .

In fact we will prove the following:

Theorem 1. *Let k and d be given positive integers, with $d \geq 3$. If $d = 3$, then there exist only finitely many connected d -valent 2-arc-transitive graphs of order kp for some prime p . If $d \geq 4$, then there exist only finitely many connected d -valent 2-arc-transitive graphs of order kp or kp^2 for some prime p .*

Note that this theorem fails for 3-valent arc-transitive graphs of order kp^2 , as shown by the existence of infinitely many 3-valent 2-arc-transitive graphs of order $6p^2$ (for p prime), obtainable as \mathbb{Z}_p^2 -covers of $K_{3,3}$; see [16, Table 1], or [7, Theorem 5.1]. It also fails for both 3- and 4-valent graphs of order kp^3 , as exhibited by the existence of infinitely many 3-valent 2-arc-transitive graphs of order $4p^3$, obtainable as \mathbb{Z}_p^3 -covers of K_4 (see [10] or [7, Theorem 4.1]), and infinitely many 4-valent 2-arc-transitive graphs of order $5p^3$, obtainable as \mathbb{Z}_p^3 -covers of K_5 (see [14, Table 1]).

Theorem 1 is a consequence of a much more general (but also more technical) theorem, the formulation of which requires some definitions.

Throughout this paper, we will let Γ be a finite connected simple undirected graph, and let $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$ be its vertex set, edge set and arc set respectively, where an *arc* is an ordered pair (u, v) of adjacent vertices. Similarly, if s is a positive integer, then an s -arc of Γ is an ordered $(s + 1)$ -tuple $(v_0, v_1, v_2, \dots, v_s)$ of vertices of Γ in which any two consecutive vertices are adjacent and any three consecutive vertices are pairwise distinct, and we will denote by $A_s(\Gamma)$ the set of all s -arcs of Γ .

The group of all automorphisms of Γ is denoted by $\text{Aut}(\Gamma)$. If this group has a single orbit on $V(\Gamma)$, or on $E(\Gamma)$, or $A(\Gamma)$, or $A_s(\Gamma)$, then Γ is said to be *vertex-transitive*, *edge-transitive*, *arc-transitive*, or *s -arc-transitive*, respectively. The term *symmetric* is synonymous with *arc-transitive*, in this context. More generally, if a subgroup G of $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, or $E(\Gamma)$, or $A(\Gamma)$, or $A_s(\Gamma)$, then we say that the graph Γ is *G -vertex-transitive*, *G -edge-transitive*, *G -arc-transitive*, or *(G, s) -arc-transitive*, respectively.

Next, for any subgroup G of $\text{Aut}(\Gamma)$, and for any vertex $v \in V(\Gamma)$, let G_v be the stabiliser $\{g \in G \mid v^g = v\}$ of v in G , and let $G_v^{\Gamma(v)}$ denote the permutation group induced by the action of G_v on the neighbourhood $\Gamma(v)$ of v . Also denote the kernel of this action by $G_v^{[1]}$. Note that a G -vertex-transitive graph Γ is G -arc-transitive if and only if $G_v^{\Gamma(v)}$ is transitive on $\Gamma(v)$, and is $(G, 2)$ -arc-transitive if and only if $G_v^{\Gamma(v)}$ is 2-transitive on $\Gamma(v)$.

A permutation group for which the stabiliser of every point is trivial is called *semiregular*. A permutation group in which every non-trivial normal subgroup is transitive is called *quasiprimitive*.

Suppose from now on that Γ is G -vertex-transitive. If the group $G_v^{\Gamma(v)}$ is permutation isomorphic to some permutation group L (for some and therefore every vertex v of Γ), then we say that G is *locally L* . Similarly, if $G_v^{\Gamma(v)}$ is a quasiprimitive permutation group, then we say that G is *locally quasiprimitive*. Also following [26], we say that a transitive permutation group L is *graph-restrictive* provided there exists a constant $c = c(L)$ such that whenever G is an arc-transitive, locally L group of automorphisms of a graph Γ , the order of the stabiliser G_v is at most $c(L)$.

If N is any subgroup of $\text{Aut}(\Gamma)$, one may construct the quotient graph Γ/N , the vertex set of which is the set of N -orbits on $V(\Gamma)$, with two such orbits adjacent in Γ/N whenever there is an edge between them in Γ . Here we note that there is a natural graph epimorphism $\varphi: \Gamma \rightarrow \Gamma/N$, mapping a vertex v to the N -orbit v^N . If φ happens to map the neighbourhood $\Gamma(v)$ of every vertex $v \in V(\Gamma)$ bijectively onto the neighbourhood of v^N in Γ/N , then we say that φ is a *N -regular covering projection* (or simply a *regular covering projection*).

We can now state our more general theorem. Here we let C_n and D_n stand respectively for the cyclic group of order n and the dihedral group of order $2n$, in their natural transitive actions on n points.

Theorem 2. *Let L be a quasiprimitive graph-restrictive permutation group of degree $d \geq 3$, with corresponding constant $c(L)$, and let k be a given positive integer, and p any prime satisfying $p \geq kc(L)$. Also suppose there exists a G -arc-transitive graph Γ of order kp^α where $\alpha = 1$ or 2 , such that $G_v^{\Gamma(v)}$ is permutation isomorphic to L , and let P be a Sylow p -subgroup of G . Then the following hold:*

- (a) *The subgroup P is normal in G , has order p^α , and acts semiregularly on $V(\Gamma)$;*
- (b) *If $k \geq 3$, $\bar{\Gamma} = \Gamma/P$ and $\bar{G} = G/P$, then the natural projection $\Gamma \rightarrow \bar{\Gamma}$ is a regular covering projection, $\bar{\Gamma}$ has order k , and \bar{G} is an arc-transitive and locally L group of automorphisms of $\bar{\Gamma}$;*
- (c) *The stabiliser G_v is isomorphic to a subgroup of $\text{Aut}(P)$;*
- (d) *The degree d of L (the valency of Γ) is prime, and one of the following holds:*
 - (i) *P is cyclic, $p \equiv 1 \pmod{2d}$, $G_v^{\Gamma(v)} \cong C_d$, $|\bar{G}| = kd$ and $\bar{G}/[\bar{G}, \bar{G}] \cong C_{2d}$;*
 - (ii) *$\alpha = 2$, P is elementary abelian of order p^2 , and $G_v^{\Gamma(v)} \cong C_d$ or D_d .*

Theorem 2 will be proved in Section 3, after some further background is given in Section 2, and Theorem 1 is proved in Section 4. Then in Section 5 we describe a means for constructing examples of symmetric graphs of order kp for a given positive integer k and variable prime p , under certain conditions, and finally, we consider the special case of symmetric cubic graphs in Section 6.

Before continuing, we comment on the assumptions made in Theorem 2 about quasiprimitivity and graph-restrictiveness of the group L .

The importance of quasiprimitivity of the group $G_v^{\Gamma(v)}$ for a G -arc-transitive graph Γ was first observed by Cheryl Praeger in [21], after noticing that such pairs (Γ, G) behave nicely with regard to taking a quotient Γ/N by a normal subgroup N of G . Local quasiprimitivity has now become a standard assumption in many applications of ‘quotienting’ techniques.

A classical topic in algebraic graph theory is the question whether the order of a vertex-stabiliser G_v for a connected d -valent G -arc-transitive graph Γ can be bounded by an absolute constant (depending only on d). A famous instance is the theorem of Tutte that gives $|G_v| \leq 48$ when $d = 3$. For larger d , the boundedness of $|G_v|$ depends not only on d , but also on the permutation group $G_v^{\Gamma(v)}$; for example, when $d = 4$ the order of G_v can be bounded by a constant provided that $G_v^{\Gamma(v)} \cong \mathbb{Z}_2^2, \mathbb{Z}_4, A_4$ or S_4 . To capture this phenomenon, the term *graph-restrictiveness* was coined by Gabriel Verret in [26].

Using Verret’s terminology, one can easily express several classical results and conjectures in a different way. For example, Tutte’s theorem says that the two transitive groups of degree 3, namely C_3 and S_3 , are graph restrictive, with corresponding constants $c(C_3) = 3$ and $c(S_3) = 48$. Similarly, it can be deduced from the work of Gardiner [13] that the alternating group A_4 and the symmetric group S_4 (both of degree 4) are graph restrictive, with corresponding constants $c(A_4) = 36$ and $c(S_4) = 2^4 3^6$.

An even stronger theorem holds, thanks to work by Richard Weiss and Vladimir Trofimov, namely that every doubly transitive group is graph-restrictive. The proof of this fact can be found by putting together pieces from many papers, but a nice summary is given in the introduction to a later paper by Weiss [30].

Also when this is taken together with another theorem proved in [29], it implies that every transitive permutation group of prime degree is graph-restrictive. Other examples of graph-restrictive groups can be found in [26, 27], and a summary of all known graph-restrictive groups is given in [17]. In particular, it is shown in [17] that if L is any transitive permutation group of degree at most 8, then L is graph-restrictive if and only if every normal subgroup of L is either transitive or semiregular.

We conclude this discussion of graph-restrictiveness by pointing out two related conjectures. The first is the ‘Weiss conjecture’, made by Richard Weiss [28, Conjecture 3.12]; this can be re-worded to say that every primitive permutation group is graph-restrictive. The second is due to Cheryl Praeger [20], and essentially states that every quasiprimitive permutation group is graph-restrictive. Note that in view of the Praeger conjecture, the condition in Theorem 2 on graph-restrictiveness might very well not be needed, since it would follow automatically from quasiprimitivity.

2. FURTHER BACKGROUND

We begin this section with a classical property of quotients of locally quasiprimitive graphs, which will be used frequently in the proofs of our theorems.

Lemma 3. [21, Section 1]. *Let Γ be a connected G -arc-transitive graph, let N be a normal subgroup of G , and take $\bar{G} = G/N$ and $\bar{\Gamma} = \Gamma/N$, and $\bar{v} = v^N$ for each vertex v of Γ . Then there is a natural (but not necessarily faithful) action of \bar{G} on $\bar{\Gamma}$ as an arc-transitive group of automorphisms. If also $G_v^{\Gamma(v)}$ is quasiprimitive and N has at least 3 orbits on $V(\Gamma)$, then N is semiregular on Γ , and the action of \bar{G} on $\bar{\Gamma}$ is faithful. Moreover, if G is locally primitive, then the natural projection $\Gamma \rightarrow \bar{\Gamma}$ is a regular covering projection, and the groups $G_v^{\Gamma(v)}$ and $\bar{G}_{\bar{v}}^{\bar{\Gamma}(\bar{v})}$ are permutation isomorphic.*

Lemma 3 has the following easy consequence:

Lemma 4. *Let Γ be a connected G -arc-transitive graph, and let N be a normal subgroup of G . If $G_v^{\Gamma(v)}$ is quasiprimitive and N_v is non-trivial, then N has at most two orbits on $V(\Gamma)$.*

We now give some other background theory that will be useful.

Lemma 5. *Suppose Γ is a connected G -arc-transitive graph of valency d , and $G_v^{\Gamma(v)}$ is a cyclic group of order d . Then G acts regularly on the arcs of Γ , and is generated by two elements of orders d and 2 , such that the element of order d generates G_v .*

Proof. First, since $G_v^{\Gamma(v)}$ acts regularly on $\Gamma(v)$ for all v , the connectivity of Γ implies that the kernel of the action of G_v on $\Gamma(v)$ is trivial, and so $G_v \cong G_v^{\Gamma(v)}$. In particular, G acts regularly on the arcs of Γ . Next, by arc-transitivity, G is generated by G_v and an element τ that interchanges v with one of its neighbours, say w . Then τ^2 stabilises the arc (v, w) , so $\tau^2 = 1$. Hence G is generated by an element of order d (generating G_v) and this element τ of order 2 . \square

Lemma 6. *Let G be a quasiprimitive permutation group. Then G contains at most two minimal normal subgroups, and its socle $M = \text{soc}(G)$ is a direct product of pairwise isomorphic simple groups. Furthermore, if G is soluble, then G is primitive of affine type; in other words, M is the only minimal normal subgroup of G , and is isomorphic to an elementary abelian group \mathbb{Z}_p^d , and then G is permutation isomorphic to a subgroup of the affine group $\text{AGL}(d, p) \cong \mathbb{Z}_p^d \rtimes \text{GL}(p, d)$ in its natural action on the vector space \mathbb{Z}_p^d .*

Proof. This follows directly from the first three paragraphs of [19, Section 3]. \square

For a prime p and a group G , let $O_p(G)$ denote the largest normal p -subgroup of G . Note that $O_p(H) \triangleleft O_p(G)$ whenever $H \triangleleft G$.

Lemma 7. *Let Γ be a connected G -arc-transitive graph, let uv be an arc of Γ , and let p be any prime. If G_v contains a non-trivial p -group C as a normal subgroup, then either $C^{\Gamma(v)} \neq 1$ or $O_p(G_{uv}^{\Gamma(v)}) \neq 1$.*

Proof. Let C be any non-trivial normal p -group of G_v , and suppose that $C^{\Gamma(v)} = 1$. Then $C \leq G_v^{[1]} \leq G_{uv}$, and since C is normal in G_v , it follows that C is normal both in G_{uv} and $G_v^{[1]}$. In particular, $O_p(G_v^{[1]})$ and $O_p(G_{uv})$ are non-trivial. Then since $G_v^{[1]} \triangleleft G_{uv}$, it follows that $O_p(G_v^{[1]}) \triangleleft O_p(G_{uv})$.

Now suppose also that $O_p(G_{uv}^{\Gamma(v)}) = 1$. Then $O_p(G_{uv})$ is a (normal) subgroup of $G_v^{[1]}$, and since $O_p(G_v^{[1]}) \triangleleft O_p(G_{uv})$, it follows that $O_p(G_{uv}) = O_p(G_v^{[1]})$. Hence $O_p(G_{uv})$ is a characteristic subgroup of G_{uv} as well as one of $G_v^{[1]}$. Then since G_{uv} is normal (of index 2) in the edge-stabiliser $G_{\{u,v\}}$ and $G_v^{[1]}$ is normal in G_v , we find that $O_p(G_{uv})$ is normal in $\langle G_v, G_{\{u,v\}} \rangle = G$, with the latter equality following from connectedness of G . But then $O_p(G_{uv})$ acts trivially on the arcs of Γ , which contradicts the fact that $O_p(G_{uv}) \neq 1$. This shows that $O_p(G_{uv}^{\Gamma(v)}) \neq 1$, as required. \square

Lemma 8. *Let Γ be a finite connected G -vertex-transitive graph. Then every simple section of G_v is also a section of $G_v^{\Gamma(v)}$. In particular, if $G_v^{\Gamma(v)}$ is soluble, then so is G_v , and if the prime p divides $|G_v|$, then also p divides $|G_v^{\Gamma(v)}|$.*

Proof. This is almost folklore, and a more general version can be found in [18]. \square

In the proof of our main theorem, we will need the following fact about the subgroups of $\text{GL}(2, p)$. We thank Pablo Spiga for offering us a proof, which uses the fact that when the prime p is congruent $\pm 1 \pmod{5}$, the group $\text{SL}(2, p)$ contains subgroups isomorphic to $\text{SL}(2, 5)$, and all such subgroups are maximal (see [22, p. 417, Ex. 7]).

Lemma 9. *Let p be a prime congruent ± 1 modulo 5, let $G = \text{GL}(2, p)$, and let T be the subgroup of order 2 in G generated by the negative identity matrix $-I_2$. Also let H be a subgroup of G isomorphic to $\text{SL}(2, 5)$, and let N be the normaliser of H in G . Then T is a characteristic subgroup of N , and of every subgroup of N containing T .*

Proof. Let Z be the centre of G . We will show first that $N = ZH$. Let $S = \text{SL}(2, p)$, and let \mathcal{H} be the set of all subgroups of G isomorphic to $\text{SL}(2, 5)$. Now consider the action of G on \mathcal{H} by conjugation. Then G is transitive on \mathcal{H} , while S has two orbits on \mathcal{H} , of equal size (since S is normal in G); see for example [22, p. 416, Ex. 2]. The stabiliser of H in G is $N_G(H) = N$, while the stabiliser of H in S is $N_S(H) = H$, since H is maximal but not normal in S . Hence by the orbit-stabiliser theorem, we have $|G| = |\mathcal{H}||N|$ and $|S| = |\mathcal{H}||H|/2$, and so $|Z| = p - 1 = |G/S| = |G|/|S| = 2|N|/|H|$. This implies $|ZH| = |Z||H|/|Z \cap H| = |Z||H|/2 = |N|$, and then since N contains both Z and H , it follows that $N = ZH$.

Next, H has just one involution, and this must be the unique involution in S , namely $-I_2$, which also lies in the central subgroup Z of $ZH = N$, and hence must be the only involution in N . It follows that the subgroup T generated by this involution is characteristic in N , and also in any subgroup of $ZH = N$ containing T . \square

We will also use the following.

Lemma 10. *Let H and K be normal subgroups of G , the orders of which are coprime. Then $HK \cap S = (H \cap S)(K \cap S)$ for every subgroup S of G .*

Proof. Clearly $(H \cap S)(K \cap S) \subseteq HK \cap S$, so it suffices to prove the reverse inclusion. Now suppose $x \in H$ and $y \in K$, with $xy \in S$. If α and β are the orders of x and y , respectively, then $\gcd(\alpha, \beta) = 1$ and it follows that there exists an integer γ with $\alpha\gamma \equiv 1 \pmod{\beta}$. Also because H and K have coprime orders, they intersect trivially, and so $[H, K] \subseteq H \cap K = 1$, which means that H and K centralise each other. In particular, x commutes with y , and therefore $y^\alpha = x^\alpha y^\alpha = (xy)^\alpha \in S$, which in turn gives $y = y^{\alpha\gamma} = (y^\alpha)^\gamma \in S$, and then also $x = (xy)y^{-1} \in S$. Thus $x \in H \cap S$ and $y \in K \cap S$, and so $xy \in (H \cap S)(K \cap S)$, as required. \square

3. PROOF OF THEOREM 2

We prove Theorem 2 in stages, beginning with the following:

Lemma 11. *Let Γ be a finite connected G -arc-transitive graph of valency $d \geq 3$, such that $G_v^{\Gamma(v)}$ is quasiprimitive. Also suppose that G has an abelian normal Sylow p -subgroup P acting semiregularly on the vertex-set of Γ , where p is an odd prime. Then the centraliser $C_G(P)$ of P in G is a direct product $J \times P$ for some normal subgroup J of G , and $C_G(P)$ acts semiregularly on $V(\Gamma)$. In particular, $\text{Aut}(P)$ contains a subgroup isomorphic to G_v . Moreover, if the valency d of Γ is prime, and $G_v^{\Gamma(v)}$ is cyclic (of order d) and also P is cyclic, then $p \equiv 1 \pmod{2d}$, each of the groups G and $\bar{G} = G/P$ is generated by two elements of orders d and 2 , and $[G, G] = C_G(P)$, $[\bar{G}, \bar{G}] \cong J$, and $G/[G, G] \cong \bar{G}/[\bar{G}, \bar{G}] \cong C_{2d}$.*

Proof. First note that since P acts semiregularly on $V(\Gamma)$, p divides $|V(\Gamma)|$.

Now let $C = C_G(P)$. Then C contains P and is normal in G (with factor group G/C isomorphic to a subgroup of $\text{Aut}(P)$). By the Schur-Zassenhaus theorem, we find that P has a complement in C , and therefore $C = J \times P$ for some subgroup J of C . Note that J is a Hall p' -subgroup of C , so J is characteristic in C and hence normal in G . Also $C_v = C \cap G_v = JP \cap G_v$, and so by Lemma 10, we find that $C_v = (J \cap G_v)(P \cap G_v) = J_v P_v = J_v$ (since $P_v = 1$).

If C_v (and therefore J_v) is non-trivial, then by Lemma 4 we know that J has at most two orbits on $V(\Gamma)$. But $J \triangleleft G$, and G is transitive on $V(\Gamma)$, so all the orbits of J have the same length, and since $|J|$ is coprime to p , it follows that the number of orbits of J on $V(\Gamma)$ is divisible by p , a contradiction. Thus $C_v = 1$, or in other words, C is semiregular on $V(\Gamma)$. Furthermore, this implies that the image of G_v in the factor group G/C is $G_v C/C \cong G_v/(G_v \cap C) = G_v/C_v \cong G_v$, and so G_v is isomorphic to a subgroup of $\text{Aut}(P)$. This proves the first part of the lemma.

Next, we suppose that the valency d is prime, and that $G_v^{\Gamma(v)} \cong C_d$, and P is cyclic, say of order p^α . By Lemma 5 we know that G is generated by an element h of order d and an element a of order 2 , and it follows that the abelianisation $G/[G, G]$ is a

quotient of the group $C_d \times C_2 \cong C_{2d}$. On the other hand, since G/C is isomorphic to a subgroup of $\text{Aut}(P) \cong \text{Aut}(C_{p^\alpha})$, which is abelian, we find that $[G, G] \leq C$, and hence also G/C is a quotient of C_{2d} . We will show that $[G, G] = C$.

Recall that $C = J \times P$, with $J \triangleleft G$, and consider the quotient G/J . We have $|G/J| = |P||G/C| = p^\alpha |G/C|$, and since $|G/C|$ divides $2d$, it follows that $|G/J|$ divides $2dp^\alpha$. Next, because P is semiregular, we know that p is coprime to the order of G_v , and in particular $d \neq p$. Also G/J is generated by Jh and Ja , the orders of which divide the primes d and 2 respectively. Now if one of these cosets were trivial, then G/J would be cyclic of order $1, 2$ or d , but then its order would be coprime to p , a contradiction. Hence Jh and Ja have orders d and 2 , so $|G/J|$ is divisible by $2d$, and therefore by $2dp^\alpha$ (again since $|J|$ is coprime to p). Thus $|G/J| = 2dp^\alpha$.

In turn this implies that $|G/C| = |G/J|/p^\alpha = 2d$, and then since $[G, G] \leq C$ and $G/[G, G]$ is a quotient of C_{2d} , we deduce that $[G, G] = C$ and $G/[G, G] = G/C \cong C_{2d}$.

Moreover, since G/C is isomorphic to a subgroup of $\text{Aut}(P) \cong \text{Aut}(C_{p^\alpha})$, which is cyclic of order $\phi(p^\alpha) = p^{\alpha-1}(p-1)$, we find that $2d = |G/C|$ divides $p^{\alpha-1}(p-1)$ and hence divides $p-1$. Thus $p \equiv 1 \pmod{2d}$.

Finally, we consider the quotient $\bar{G} = G/P$. This is generated by the images Ph and Pa , which have orders d and 2 (since the latter are both coprime to $|P|$). Also $[\bar{G}, \bar{G}] = [G/P, G/P] \cong [G, G]P/P = CP/P = C/P \cong J$, and it immediately follows that $\bar{G}/[\bar{G}, \bar{G}] \cong (G/P)/(C/P) \cong G/C \cong G/[G, G] \cong C_{2d}$. \square

Lemma 12. *Let Γ be a finite connected G -arc-transitive graph with valency $d \geq 3$, and suppose $G_v^{\Gamma(v)}$ is quasiprimitive. Also suppose that the order of Γ is kp or kp^2 for some positive integer k and some prime p which divides neither k nor $|G_v^{\Gamma(v)}|$, and that G has a normal Sylow p -subgroup P . Then parts (a) and (c) of Theorem 2 hold, and if the valency d is prime, then so does part (b).*

Proof. Suppose $|V(\Gamma)| = kp^\alpha$, so that $\alpha = 1$ or 2 . Because p is coprime to $|G_v^{\Gamma(v)}|$, we know from Lemma 8 that p is coprime to $|G_v|$, and it follows that $P_v = 1$, or in other words, P is semiregular on $V(\Gamma)$. Moreover, since $|G| = kp^\alpha |G_v|$ and p is coprime to both k and $|G_v|$, we find that $|P| = p^\alpha$. In particular, P is abelian (since $\alpha \leq 2$), so the conditions of Lemma 11 are fulfilled, and therefore G_v is isomorphic to a subgroup of $\text{Aut}(P)$. These observations prove parts (a) and (c).

Next suppose $k \geq 3$. Then the order of the quotient graph $\bar{\Gamma} = \Gamma/P$ is equal to the number of P -orbits on $V(\Gamma)$, which is $|V(\Gamma)|/|P| = k$. Also since $k \geq 3$ and $G_v^{\Gamma(v)}$ is quasiprimitive, an application of Lemma 3 (with $N = P$) shows that \bar{G} acts faithfully and arc-transitively on $\bar{\Gamma}$. Moreover, if the valency d is prime then G is locally primitive, and it follows (by Lemma 3) that the natural projection $\Gamma \rightarrow \bar{\Gamma}$ is a regular covering projection, and $G_v^{\Gamma(v)}$ and $\bar{G}_{\bar{v}}^{\bar{\Gamma}(\bar{v})}$ are permutation isomorphic, so that \bar{G} is locally $G_v^{\Gamma(v)}$. This proves part (b). \square

Now to complete the proof of Theorem 2, all we need to do is prove part (d), which includes showing that the valency d is prime. We will use the fact that $|P| = p$ or p^2 , from which it follows that P is cyclic or elementary abelian of rank 2.

We proceed by dealing with two special cases.

Lemma 13. *If $G_v^{\Gamma(v)}$ is abelian, then part (d) of Theorem 2 holds.*

Proof. Since $G_v^{\Gamma(v)}$ is abelian, it is regular on $\Gamma(v)$, and since it is also quasiprimitive, it must be cyclic of prime order. Thus d is prime, and $G_v^{\Gamma(v)} \cong C_d$. Moreover, since $G_v^{\Gamma(v)}$ is regular, it follows from connectivity of Γ that the kernel $G_v^{[1]}$ is trivial, and hence $G_v \cong G_v^{\Gamma(v)}$. Finally, if P is cyclic, then part (d)(i) of Theorem 2 holds by Lemma 12; and on the other hand, if P is not cyclic, then $\alpha = 2$ and P is elementary abelian, so part (d)(ii) holds. \square

Lemma 14. *If the Sylow subgroup P of G is cyclic, then part (d) of Theorem 2 holds.*

Proof. Since P is cyclic of prime power order, $\text{Aut}(P)$ is cyclic, and hence G_v is cyclic, by part (c). In turn this implies that $G_v^{\Gamma(v)}$ is cyclic, and Lemma 13 applies. \square

For the rest of the proof we may now assume that $\alpha = 2$, and P is elementary abelian of order p^2 . Also it suffices to show that d is prime, and $G_v^{\Gamma(v)} \cong C_d$ or D_d . The latter holds automatically when $d = 3$, so we can also assume that $d \geq 4$.

Now since $P \cong \mathbb{Z}_p^2$, we know $\text{Aut}(P)$ is isomorphic to $\text{GL}(2, p)$, and by part (c) it follows that G_v is isomorphic to a subgroup of $\text{GL}(2, p)$. In view of this, we can think of G_v as a subgroup of $\text{GL}(2, p)$. We will consider the intersection

$$H = G_v \cap \text{SL}(2, p).$$

This is a normal subgroup of G_v , with cyclic quotient, since

$$G_v/H = G_v/(G_v \cap \text{SL}(2, p)) \cong G_v \text{SL}(2, p)/\text{SL}(2, p) \leq \text{GL}(2, p)/\text{SL}(2, p) \cong C_{p-1}.$$

Also if H is contained in $G_v^{[1]}$, then $G_v^{\Gamma(v)} \cong G_v/G_v^{[1]} \cong (G_v/H)/(G_v^{[1]}/H)$, which is a quotient of G_v/H and therefore cyclic, and then part (d) follows from Lemma 13.

We may therefore assume that H is not contained in $G_v^{[1]}$, and hence that $H^{\Gamma(v)}$ is a non-trivial normal subgroup of $G_v^{\Gamma(v)}$. Moreover, since $G_v^{\Gamma(v)}$ is quasiprimitive, this implies that $H^{\Gamma(v)}$ is transitive on $\Gamma(v)$.

On the other hand, H is a subgroup of $\text{SL}(2, p)$, with order coprime to p , and so it follows from the classification of subgroups of 2-dimensional special linear groups [22, Theorem 6.17] that H is isomorphic to one of the following:

- (1) a cyclic group;
- (2) a metacyclic group $\langle x, y \mid x^n = y^2, y^{-1}xy = x^{-1} \rangle$ of order $2n$;

- (3) the group \hat{S}_4 of order 48 with a unique involution τ such that $\langle \tau \rangle$ is the centre of \hat{S}_4 , and $\hat{S}_4/\langle \tau \rangle \cong S_4$;
- (4) the special linear group $\text{SL}(2, 3)$;
- (5) the special linear group $\text{SL}(2, 5)$, in cases where $p \equiv \pm 1 \pmod{5}$.

This allows us to prove the following:

Lemma 15. *The valency d is at least 5. Moreover, the group $H^{\Gamma(v)} \cong H/(H \cap G_v^{[1]})$ is either cyclic or dihedral, or isomorphic to a quotient of A_4 , S_4 or A_5 .*

Proof. Suppose first that H has odd order. Then H is cyclic (since all the groups in cases (2) to (5) above have even order), and therefore $|H^{\Gamma(v)}|$ is cyclic of odd order. By transitivity of $H^{\Gamma(v)}$ on $\Gamma(v)$, it follows that the valency $d = |\Gamma(v)|$ is odd. Since $d \geq 4$, it follows that $d \geq 5$.

Hence we may assume that H has even order. In that case, H must contain the centre T of $\text{SL}(2, p)$, generated by the unique involution $-I_2$ in $\text{SL}(2, p)$. In particular, the latter is the only involution in H , so T is characteristic in H , and hence normal in G_v . On the other hand, $G_v^{\Gamma(v)}$ is quasiprimitive, and so contains no non-trivial normal subgroups of order less than $d = |\Gamma(v)|$. It follows that $T^{\Gamma(v)} = 1$, or equivalently, $T \leq G_v^{[1]}$. Then by Lemma 7, we find that $O_2(G_v^{\Gamma(v)}) \neq 1$. If $d = 4$, then $G_v^{\Gamma(v)}$ is permutation isomorphic to A_4 or S_4 in their natural actions on 4 points (these being the only quasiprimitive permutation groups of degree 4). Then $G_{uv}^{\Gamma(v)} \cong C_3$ or S_3 . However, neither of these two groups contains a non-trivial normal 2-subgroup. This contradiction shows that $d \geq 5$.

Moreover, since $T \leq H \cap G_v^{[1]}$, we know that

$$H^{\Gamma(v)} \cong H/(H \cap G_v^{[1]}) \cong (H/T)/((H \cap G_v^{[1]})/T),$$

and so $H^{\Gamma(v)}$ is a quotient of H/T . By inspection of the groups in cases (1) to (5), we see that H/T is cyclic in case (1), dihedral of order $2n$ in case (2), S_4 in case (3), $\text{PSL}(2, 3) \cong A_4$ in case (4), and $\text{PSL}(2, 5) \cong A_5$ in case (5), and the rest follows. \square

We complete the proof of Theorem 2 by considering the cases (1) to (5) above in more detail.

Lemma 16. *Part (d) of Theorem 2 holds in cases (1) and (2).*

Proof. In these two cases, H is soluble, and then since $G_v/H = G_v/(G_v \cap \text{SL}(2, p))$ is cyclic, both G_v and $G_v^{\Gamma(v)}$ are soluble too. By Lemma 6, we find that the socle $M = \text{soc}(G_v^{\Gamma(v)})$ is an elementary abelian group, of order q , say, and that $G_v^{\Gamma(v)}$ is permutation isomorphic to a subgroup of $\text{AGL}(1, q)$, so $d = q$, and M is the only minimal normal subgroup of $G_v^{\Gamma(v)}$. Hence in particular, $M = \text{soc}(G_v^{\Gamma(v)})$ is a subgroup of $H^{\Gamma(v)}$. But $H^{\Gamma(v)}$ is a quotient of H/T , and so is either cyclic or dihedral, and therefore M is cyclic or dihedral, which implies that $M \cong C_q$ or $C_2 \times C_2$. In the

latter case, however, $d = q = 4$, which is impossible by Lemma 15. Thus $M \cong C_q$, and since M is elementary abelian, it follows that $d = q$ is prime.

It remains to show that $G_v^{\Gamma(v)} \cong C_q$ or D_q .

Let C be the largest cyclic subgroup of H , which has index 1 or 2 in H . In fact $C = H$ in case (1), or the unique subgroup $\langle x \rangle$ of order n and index 2, in case (2). In each case, C is a characteristic subgroup of H , and hence normal in G_v , and it follows that $C^{\Gamma(v)}$ is a normal subgroup of $G_v^{\Gamma(v)}$, of index at most 2 in $H^{\Gamma(v)}$. Then because $G_v^{\Gamma(v)}$ is quasiprimitive, $C^{\Gamma(v)}$ is transitive and hence regular. But now $C^{\Gamma(v)}$ must be a minimal normal subgroup of $G_v^{\Gamma(v)}$, and in particular, $C^{\Gamma(v)} = \text{soc}(G_v^{\Gamma(v)}) \cong C_q$.

Next, since C is a cyclic subgroup of $\text{SL}(2, p)$ with order $|C|$ dividing $|H|$ and hence coprime to p , we find that C is generated by some diagonal matrix of the form

$$X = \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix}.$$

A direct computation shows that any matrix $A \in \text{GL}(2, p)$ that conjugates X to some power of X is either diagonal or of the form

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}.$$

In both cases, A^2 is a diagonal matrix that commutes with X . Then since C is normal in G_v , it follows that the square of every element of G_v centralises C , and hence the square of every element of $G_v^{\Gamma(v)}$ centralises $C^{\Gamma(v)}$. But $G_v^{\Gamma(v)}$ is permutation isomorphic to a subgroup of $\text{AGL}(1, q)$, with $C^{\Gamma(v)} = \text{soc}(G_v^{\Gamma(v)}) \cong C_q$ as a regular normal subgroup, and so it follows that $G_v^{\Gamma(v)}$ is isomorphic to C_q or to the dihedral group D_q , as required. \square

Lemma 17. *Cases (3) and (4) are impossible.*

Proof. In these cases, $H \cong \hat{S}_4$ or $\text{SL}(2, 3)$, so again H is soluble, and G_v and $G_v^{\Gamma(v)}$ are soluble too, and then by Lemma 6, the socle $M = \text{soc}(G_v^{\Gamma(v)})$ is elementary abelian of order d , and is the only minimal normal subgroup of G_v . In particular, $H^{\Gamma(v)}$ contains M , of order $d \geq 5$. On the other hand, $H^{\Gamma(v)}$ is isomorphic to a quotient of A_4 or S_4 , and so is isomorphic to one of S_4 , A_4 , S_3 , C_3 and C_2 , but none of these groups has an elementary abelian subgroup of order at least 5, a contradiction. \square

This leaves us with just case (5) to check, with $H \cong \text{SL}(2, 5)$.

Lemma 18. *Case (5) is impossible.*

Proof. In this case $p \equiv \pm 1 \pmod{5}$, and $H \cong \text{SL}(2, 5)$, so $H/T \cong \text{PSL}(2, 5) \cong A_5$. Since $G_v^{\Gamma(v)} \cong G_v/G_v^{[1]}$ has no normal subgroup of order 2, we know that $T = \langle -I_2 \rangle$ is a subgroup of $G_v^{[1]}$, and hence also of the arc-stabiliser G_{uv} . Next, G_{uv} is a subgroup

of G_v , so normalises H , and is a subgroup of $\text{GL}(2, p)$. By Lemma 9 with $N = G_{uv}$, we conclude that T is characteristic in G_{uv} . Then since G_{uv} is normal (of index 2) in the edge-stabiliser $G_{\{u,v\}}$, it follows that T is normal in $G_{\{u,v\}}$, and hence also normal in the group $G^* = \langle G_v, G_{\{u,v\}} \rangle$. But Γ is connected and G is transitive on the arcs of Γ , so $\langle G_v, G_{\{u,v\}} \rangle = G$, and therefore T is normal in G . On the other hand, T is contained in H and hence in G_v , and so cannot be normal in G , for otherwise it would fix every vertex of Γ . This contradiction completes the proof. \square

4. PROOF OF THEOREM 1

Theorem 1 is an almost immediate consequence of Theorem 2.

For suppose $d \geq 4$, and Γ is a connected d -valent 2-arc-transitive graph of order kp or kp^2 for some prime p , and let G be a 2-arc-transitive group of automorphisms of Γ . Then $L = G_v^{\Gamma(v)}$ is 2-transitive and therefore graph-restrictive, so Theorem 2 applies, and we find that if $p \geq kc(L)$ then $G_v^{\Gamma(v)} \cong C_d$ or D_d . But the latter are not 2-transitive, so this implies $p < kc(L)$, and hence there are only finitely many possibilities for p , and so finitely many possibilities for Γ .

Similarly, if $d = 3$ and Γ has order kp , then only case (i) of part (d) of Theorem 2 is possible, but in that case $G_v^{\Gamma(v)} \cong C_d$ when $p \geq kc(L)$, and again we find $p < kc(L)$.

We will consider the 3-valent case further in Section 6.

5. CONSTRUCTIONS

In this section, we consider what happens in case (i) of part (d) of Theorem 2, in more detail. Before doing this, we describe a generic construction of arc-transitive graphs, which is often attributed to Subidussi.

Given a group G , a core-free subgroup H of G , and an element $a \in G \setminus H$ such that $a^2 \in H$, we may construct a graph $\Gamma(G, H, a)$ with vertex-set $(G : H) = \{Hg : g \in G\}$, and with two (right) cosets Hx and Hy adjacent if and only if $xy^{-1} \in HaH$.

Note that right multiplication of cosets by elements of G gives rise to an arc-transitive and faithful action of G on $\Gamma(G, H, a)$. The stabiliser of the vertex H in the group G is H , and multiplication by the element a swaps the vertex H with its neighbour Ha , and the stabiliser in G of the arc (H, Ha) is the subgroup $H \cap H^a$. Thus $\Gamma(G, H, a)$ is G -arc-transitive, with order $|G : H|$ and valency $|H : H \cap H^a|$.

The importance of this construction is reflected in the fact that every G -arc-transitive graph Λ is isomorphic to $\Gamma(G, G_v, a)$, where v is an arbitrary vertex of Λ and a is an arbitrary element of G swapping v with a neighbour of v .

Also we note that two such graphs $\Gamma(G_1, H_1, a_1)$ and $\Gamma(G_2, H_2, a_2)$ are isomorphic whenever there is a group isomorphism $\varphi : G_1 \rightarrow G_2$ mapping H_1 to H_2 and a_1 to a_2 . On the other hand, is it sometimes possible for two such graphs to be isomorphic in

other cases — even when the groups G_1 and G_2 are not isomorphic — since it can happen that a graph admits more than one arc-transitive group action.

We can use the above construction to produce symmetric covers of a given arc-transitive graph, as in the proof of the following.

Theorem 19. *Let d be an odd prime, let k be an integer with $k \geq 3$, and let p be any prime such that p does not divide k , and $p \equiv 1 \pmod{2d}$. Also suppose that \bar{G} is a group of order kd such that $\bar{G}/[\bar{G}, \bar{G}] \cong C_{2d}$, and that $\bar{\Gamma}$ is a d -valent \bar{G} -arc-regular graph of order k . Then there exists at least one and at most $d-1$ pairwise non-isomorphic \mathbb{Z}_p -regular covering graphs Γ for $\bar{\Gamma}$ such that the group \bar{G} lifts along the corresponding covering projection $\Gamma \rightarrow \bar{\Gamma}$. Moreover, \bar{G} is generated by elements h and a such that $h^d = a^2 = 1$, with h generating the stabiliser of some vertex \bar{v} , and a interchanging \bar{v} with one of its neighbours, and then each of these covering graphs is isomorphic to a coset graph $\Gamma(G, \langle zh \rangle, a)$, where G is a semi-direct product $C_p \rtimes \bar{G}$, and z is a generator for the normal subgroup C_p , with $z^a = z^{-1}$ and $z^h = z^\zeta$ for some primitive d -th root ζ of 1 mod p .*

Proof. First, let Λ be any connected G -arc-regular graph of order n and valency d , with vertex-stabiliser $G_v (\cong C_d)$. Then G has order dn , and is generated by an element of order d stabilising a vertex and inducing a d -cycle on its neighbours, and an element of order 2 that interchanges the vertex with one of its neighbours. Thus G is a homomorphic image of the free product $\mathcal{U} = C_2 * C_d = \langle x, y \mid x^2 = y^d = 1 \rangle$, which is a universal group for such arc-regular actions (for valency d). Moreover, if $\eta: \mathcal{U} \rightarrow G$ is the corresponding epimorphism then Λ is isomorphic to the graph $\Gamma(G, \langle y^n \rangle, x^n)$, and in particular, the order of Λ is $|G: \langle y^n \rangle| = |\mathcal{U}: \langle N, y \rangle| = |\mathcal{U}: N|/d$.

Now take the given d -valent \bar{G} -arc-regular graph $\bar{\Gamma}$ of order k , with generators h and a for $\bar{\Gamma}$ being the images of y and x , and suppose $\wp: \Gamma \rightarrow \bar{\Gamma}$ is a \mathbb{Z}_p -regular covering projection for $\bar{\Gamma}$, such that \bar{G} lifts along \wp . If G is the lift of \bar{G} along \wp , then $G \cong \mathcal{U}/N$ and $\bar{G} \cong \mathcal{U}/K$ for normal subgroups K and N of \mathcal{U} , such that K has index kd in \mathcal{U} , and N is contained in K , with quotient K/N being cyclic of order p . To make things easier, we might as well take $G = \mathcal{U}/N$ and $\bar{G} = \mathcal{U}/K$.

Since $|G| = kpd$, and p is coprime to both k and d , we find that $P = K/N$ is a cyclic normal Sylow p -subgroup of G , and so by Lemma 11, we have $C_G(P) = [G, G] = J \times P$ for some normal subgroup J of G . Also by the Schur-Zassenhaus theorem, we know that P has a complement in G , and because $G/P \cong (\mathcal{U}/N)/(K/N) \cong \mathcal{U}/K = \bar{G}$ (and more importantly, the relations satisfied in G/P by the images of the generators of \mathcal{U} are the same as those satisfied by the images in \bar{G}), we can take this complement to be $\bar{G} = \langle h, a \rangle$. Thus G is isomorphic to a semi-direct product $C_p \rtimes \bar{G}$.

Next, the generators a and h of \bar{G} induce automorphisms of P , of orders 2 and d , respectively, since they do not lie in $[G, G] = C_G(P)$. Moreover, since -1 is the only unit of order 2 mod p , it follows that conjugation by a inverts every element of P , while conjugation by h is exponentiation by some primitive d -th root ζ of 1 mod p .

The value of ζ completely determines the structure of G , while on the other hand, the covering graph Γ is determined by the choice of images of x and y in G .

We can take the images of x and y in G as z_1a and z_2h where $z_1, z_2 \in P$. These have orders 2 and d , since $(z_1a)^2 = z_1z_1^a = z_1z_1^{-1} = 1$, and $(z_2h)^d = z_2^{1+\zeta+\zeta^2+\dots+\zeta^{d-1}}h^d = 1$, because $(1-\zeta)(1+\zeta+\dots+\zeta^{d-1}) \equiv 1-\zeta^d \equiv 0 \pmod{p}$ but $1-\zeta \not\equiv 0 \pmod{p}$. Also if we conjugate z_1a and z_2h by any element u of P , then we get

$$u^{-1}(z_1a)u = u^{-1}z_1u^{-1}a = z_1u^{-2}a \quad \text{and} \quad u^{-1}(z_2h)u = u^{-1}z_2u^{\zeta^{-1}}h = z_2u^{-1+\zeta^{-1}}h.$$

Since p is odd, we can choose u so that $u^2 = z_1$, and thereby assume that the image of x is a itself, and then we can take $z = z_2u^{-1+\zeta^{-1}}$, so that the image of y is zh . Note that z is non-trivial, since a and h generate \bar{G} (rather than G), and in particular, z is a generator for P .

It follows that the covering graph Γ is completely determined by ζ , and since there are $d-1$ choices for ζ , there are at most $d-1$ possibilities for Γ , as required.

Finally, we show that every choice of ζ gives rise to such a covering Γ . To do this, we simply take G as the semi-direct product $C_p \rtimes_{\zeta} \bar{G}$ given by ζ , and let z be any generator of the normal subgroup C_p , so that $z^a = z^{-1}$ and $z^h = z^{\zeta}$. Then a and zh are elements of orders 2 and d in G , and so we can construct the coset graph $\Gamma(G, \langle zh \rangle, a)$ in the usual way. Clearly this has $\Gamma(\bar{G}, \langle h \rangle, a) \cong \bar{\Gamma}$ as a quotient, and so all we have to do is prove that a and zh generate G . But now

$$[a, zh] = ah^{-1}z^{-1}azh = ah^{-1}z^{-2}ah = az^{-2\zeta}h^{-1}ah = z^{2\zeta}ah^{-1}ah = z^{2\zeta}[a, h],$$

and $[a, h]$ centralises z (by the definition of $C_p \rtimes_{\zeta} \bar{G}$), so if $[a, h]$ has order m in \bar{G} , then $[a, zh]^m = (z^{2\zeta}[a, h])^m = z^{2\zeta m}$, which is non-trivial (since m divides $|\bar{G}| = kd$ and hence is coprime to p), and therefore $[a, zh]^m$ generates the normal subgroup C_p .

This completes the proof. \square

We note that it is sometimes possible that different choices for ζ give isomorphic covering graphs. For example, suppose $\Gamma = \Gamma(G, \langle zh \rangle, a)$ is the covering graph given by a particular value of ζ , and there exists an automorphism of G that inverts each of the generators a and zh for G . Then replacing zh by $(zh)^{-1}$, we obtain a graph which is isomorphic to Γ . But $(zh)^{-1} = h^{-1}z^{-1} = z^{-\zeta}h^{-1}$, and conjugation by h^{-1} is exponentiation by ζ^{-1} , so it follows that ζ^{-1} gives the same graph Γ .

6. THE 3-VALENT (CUBIC) CASE

In the special case where the valency d is 3, we know from Theorem 1 that for any fixed positive integer k , there exist only finitely many connected 2-arc-transitive cubic graphs of order kp where p is prime. But Theorems 2 and 19 gives us more detailed information.

Let Γ be a connected symmetric cubic graph of order kp where k is a given even positive integer, and p is a variable prime such that $p \geq 48k$, and let G be any

arc-transitive group of automorphisms of Γ . Then by part (d)(i) of Theorem 2 with $L = C_3$ or S_3 , we know that G has a cyclic normal Sylow p -subgroup P which acts semiregularly on Γ , and that Γ has a quotient $\bar{\Gamma}$ on which $\bar{G} = G/P$ acts arc-transitively. Also the stabiliser G_v is isomorphic to a subgroup of $\text{Aut}(P)$ and is therefore cyclic. In particular, G acts regularly on the arcs of Γ , with G_v inducing C_3 on $\Gamma(v)$. Moreover, \bar{G} has order $3k$, and $\bar{G}/[\bar{G}, \bar{G}] \cong C_6$, and $p \equiv 1 \pmod{6}$.

In this case, $|G| = 3|V(\Gamma)| = 6p$, and by Theorem 19, \bar{G} is generated by elements h and a such that $h^3 = a^2 = 1$, and then G is isomorphic to the semi-direct product $C_p \rtimes_{\lambda} \bar{G}$, where λ is one of the two non-trivial cube roots of 1 in \mathbb{Z}_p , and h and a conjugate a generator z of the normal subgroup C_p to z^{λ} and z^{-1} respectively.

Note that there are just two possibilities for λ , and each is the inverse (or square) of the other. Also these two choices for λ give non-isomorphic graphs, unless there exists an automorphism of the group \bar{G} that inverts each of h and a , which happens if and only if $\bar{\Gamma}$ admits a 2-arc-regular group of automorphisms; see [6] or [9].

Note also that if there is no finite group \bar{G} of order $3k$ generated by two elements of orders 2 and 3 and with commutator subgroup $[\bar{G}, \bar{G}]$ of index 6 in \bar{G} , then there can be no connected symmetric cubic graph of order kp for any prime $p \geq 48k$.

We can now apply this information to small values of k .

6.1. Symmetric cubic graphs of order $2p$. Here $k = 2$, and the quotient graph $\bar{\Gamma}$ considered above is K_2 . However, by adjusting the definition of the quotient given in Section 1 in a way that allows multiple edges in the quotient (see [15] for details), one can view the quotient as the cubic dipole, with two vertices and three edges joining them. Although this $\bar{\Gamma}$ is not a simple graph, we can still view every large connected symmetric cubic graph of order $2p$ where p is prime as a cover of $\bar{\Gamma}$. In particular, $\bar{\Gamma}$ admits a 2-arc-regular group of automorphisms.

Hence for each such prime $p \geq 96$ with $p \equiv 1 \pmod{6}$, there is exactly one arc-regular connected cubic group of order $2p$, and its automorphism group is a semi-direct product of C_p by C_3 . Many of these graphs appear in the lists of symmetric cubic graphs given in [5] and [4], for orders up to 768 and 10000 respectively.

On the other hand, if $p < 96$, then every such Γ has order less than 192 and so appears in [5]. Apart from 1-arc-regular examples with small $p \equiv 1 \pmod{6}$, but $p \neq 7$, there are just four such Γ , namely the 2-arc-regular graph F004 $\cong K_4$ (with $p = 2$), the 3-arc-regular graphs F006 $\cong K_{3,3}$ ($p = 3$), the 3-arc-regular Petersen graph F010 ($p = 5$), and the 4-arc-regular Heawood graph F014 ($p = 7$).

In summary, every connected symmetric cubic graph of order $2p$ for some prime p is either a uniquely determined 1-arc-regular cubic graph (with $p \equiv 1 \pmod{6}$ and $p \neq 7$), or one of the four small exceptions (K_4 , $K_{3,3}$, the Petersen graph or the Heawood graph). These are the same graphs as the ones found by Cheng and Oxley in [3] using a different approach.

6.2. Symmetric cubic graphs of order $4p$. Here $k = 4$, but the only group of order 12 that can be generated by two elements of orders 2 and 3 is the alternating group A_4 , and in this group, the commutator subgroup has index 3, not 6. Hence there are no such graphs with $p \geq 48k = 192$. For smaller p , all the graphs appear in the census [5]. Again there are just four possibilities, namely the 3-cube $Q_3 \cong \text{F008}$ (which is 2-arc-regular, with $p = 2$), the dodecahedral graph F020A and the canonical double cover F020B of the Petersen graph (which are 2- and 3-arc-regular respectively, with $p = 5$), and the 3-arc-regular Coxeter graph F028 (with $p = 7$). This was shown also in [10, Section 6], by other means.

6.3. Symmetric cubic graphs of order $6p$. Here $k = 6$, and we have just one connected symmetric cubic graph of order k , namely $K_{3,3}$, which is 3-arc-transitive. Hence we have an infinite family of 1-arc-regular cubic graphs of order $6p$, one for each large prime $p \equiv 1 \pmod{6}$. The only other graphs that arise in this case have order at most $48k^2 = 1728$, and so appear in the extended census [4]. Apart from 1-arc-regular examples with small $p \equiv 1 \pmod{6}$, there are just three such Γ , namely the 3-arc-regular Pappus graph F018 (with $p = 3$), Tutte's 8-cage F030 (which is 5-arc-regular, with $p = 5$), and the 4-arc-regular Sextet graph $S(17) \cong \text{F102}$ (with $p = 17$). This classification was achieved also in [10, Section 5], by different means. (Incidentally, there is a typographic error in the introduction of [10], where a claim is made about an infinite family of cubic 2-arc-regular graphs of order $6p$; the order should be $6p^2$ (not $6p$).)

6.4. Symmetric cubic graphs of order $8p$. Here $k = 8$, and we have just one connected symmetric cubic graph of order k , namely the cube graph Q_3 , which is 2-arc-transitive. Hence we have an infinite family of 1-arc-regular cubic graphs of order $8p$, one for each large prime $p \equiv 1 \pmod{6}$. The only other graphs that arise in this case have order at most $48k^2 = 3072$, and so all of them appear in the extended census [4]. Apart from 1-arc-regular examples with small $p \equiv 1 \pmod{6}$, there are just five such graphs, namely the 2-arc-regular graphs F016 ($p = 2$) and F024 ($p = 3$), the 3-arc-regular graph F040 ($p = 5$), and the graphs F056B and F056C (which are 2- and 3-arc-regular respectively, with $p = 7$). This classification was achieved also in [12], by different means.

6.5. Symmetric cubic graphs of order $10p$. Since there is no group of order 30 generated by two elements of orders 2 and 3, there are only finitely many connected symmetric cubic graphs of order $10p$ for p prime. Moreover, since $p < 48k = 480$ for these, all such graphs appear in the extended census [4]; but in fact they all appear in [5]. Again there are just five possibilities, namely the dodecahedral graph F020A and the canonical double cover F020B of the Petersen graph (which are 2- and 3-arc-regular respectively, with $p = 2$), Tutte's 8-cage F030 (which is 5-arc-regular, with $p = 3$), the graph F050 (which is a 2-arc-regular Cayley graph for the group

$C_5 \text{ wr } S_2$, with $p = 5$) and the 3-arc-regular Coxeter-Frucht graph F110 ($p = 11$). This classification was achieved also in [11], by other means.

6.6. Symmetric cubic graphs of order $12p$. Since there is no group of order 36 generated by two elements of orders 2 and 3, there are only finitely many connected symmetric cubic graphs of order $12p$ for p prime. Moreover, since $p < 48k = 576$ for these, all such graphs appear in the extended census [4]. It is easily checked that there are just four possibilities, namely the 2-arc-regular graphs F024 ($p = 2$), F060 ($p = 5$) and F084 ($p = 7$), and the 4-arc-regular graph F204 ($p = 17$), all of which appear in [5]. This classification appears to be new.

6.7. Symmetric cubic graphs of order $14p$. Here $k = 14$, and there is just one possibility for $\bar{\Gamma}$, namely the Heawood graph F014, but this is 4-arc-transitive and admits two arc-regular actions, one via the group $C_7 \rtimes_2 C_6$ and another via $C_7 \rtimes_4 C_6$, but admits no 2-arc-regular action. It follows that for every large prime $p \equiv 1 \pmod{6}$, there are two non-isomorphic arc-regular connected cubic graphs of order $14p$. All other graphs that arise in this case have order at most $48k^2 = 9408$, and so appear in the extended census [4]. Apart from other pairs of 1-arc-regular examples with $p \equiv 1 \pmod{6}$ for $p > 7$, there are just six such graphs, and all of them appear in [5]. The exceptions are the 3-arc-regular Coxeter graph F028 (with $p = 2$), the 1-arc-regular graphs F042 (with $p = 3$) and F098B (with $p = 7$), the 2-arc-regular graph F098B (also with $p = 7$), and the graphs F182C and F182D (which are 2- and 3-arc-regular, with $p = 13$). This classification seems to be new, as well.

Note that all arc-transitive abelian regular covers of the graphs $K_{3,3}$ and Q_3 and the Heawood graph (encountered in the cases $k = 6, 8$ and 14 above) are described in the papers [7, 8].

6.8. Symmetric cubic graphs of order kp for larger k . For slightly larger values of k , some more sophisticated arguments can be used to deduce the existence of a cyclic normal Sylow p -subgroup for many values of p less than $48k$. Even without going into those, we know the following:

When $k = 16$, the graph Γ is either a uniquely determined 1-arc-regular cubic graph (with $p \equiv 1 \pmod{6}$) or one of a small finite list of exceptions, which includes the 2-arc-regular graphs F032, F048 and F112B, and the 3-arc-regular graphs F080 and F112B.

When $k = 18$, the graph Γ is either a uniquely determined 1-arc-regular cubic graph (with $p \equiv 1 \pmod{6}$), or one of a small finite list of exceptions, which includes the 2-arc-regular graph F054, and the 5-arc-regular graphs F090 and F234B.

When $k = 20$, the graph Γ is one of only a small finite number of possibilities, which include the 2-arc-regular graph F060, F220A and F220B, the 3-arc-regular graphs F040 and F220C, and the 4-arc-regular graph F620.

Moreover, it is not difficult to obtain the theorem below.

Theorem 20. *There are infinitely many (even) values of k for which there are only finitely many connected symmetric cubic graphs of order kp . In particular, this is true for all k of the form 2ℓ where ℓ is a prime congruent to 5 mod 6.*

Proof. Let $k = 2\ell$ where ℓ is as given. By Theorem 2, in order for there to exist infinitely many such graphs, there must be a finite group \bar{G} of order $3k$ generated by two elements a and h of orders 2 and 3, with commutator subgroup $\bar{G}' = [\bar{G}, \bar{G}]$ of order ℓ and index 6. Then since ℓ is prime, \bar{G}' is cyclic. Also the generator h of order 3 for \bar{G} must centralise \bar{G}' (since $\ell \not\equiv 1 \pmod{3}$), and it follows that \bar{G} has a cyclic normal subgroup of order 3ℓ and index 2. This contains subgroups of orders 3 and ℓ that are characteristic in \bar{G}' and hence normal in \bar{G} , and then factoring out the characteristic subgroup \bar{H} of order 3 gives a dihedral quotient \bar{G}/\bar{H} of order 2ℓ . But on the other hand, since \bar{H} contains the generator h of order 3, this quotient \bar{G}/\bar{H} is generated by the image of the involuntary element a , which is clearly impossible. \square

Finally, we note that this argument works also for other values of k for which there exists no finite group of order $3k$ generated by two elements of orders 2 and 3, with commutator subgroup of index 6. For small k , checking for the existence of groups with the required properties is an easy exercise using MAGMA [1].

Hence, for example, there are only finitely many connected symmetric cubic graphs of order kp for p prime when $k = 4, 10, 12, 20, 22, 28, 30, 34, 36, 40, 44, 46, 52, 58, 60, 66, 68, 70, 76, 80, 82, 84, 88, 90, 92, 94$ or 100 ; and on the other hand, there is an infinite family of such graphs whenever $k = 2, 6, 8, 14, 16, 18, 24, 26, 32, 38, 42, 48, 50, 54, 56, 62, 64, 72, 74, 78, 86, 96$ or 98 .

ACKNOWLEDGEMENTS

The authors acknowledge the use of MAGMA [1] in testing various matters considered in this paper.

The first author was generously supported by a James Cook Fellowship from the Royal Society of New Zealand, and a grant from the N.Z. Marsden Fund.

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